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Waves past porous structures in a two-layer fluid

S.R. MANAM and T. SAHOO[1]
CMLA, ENS de Cachan, 61 Avenue du President Wilson, Cachan, 94235 Cedex France. (srinimanam@rediffmail.com,); [1]Department of Ocean Engineering and Naval Architecture, IIT, Kharagpur 721 302, India. (tsahoo@naval.iitkgp.ernet.in)

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Abstract. Havelock's type of expansion theorems, for an integrable function having a single discontinuity point in the domain where it is defined, are utilized to derive analytical solutions for the radiation or scattering of oblique water waves by a fully extended porous barrier in both the cases of finite and infinite depths of water in two-layer fluid with constant densities. Also, complete analytical solutions are obtained for the boundary-value problems dealing with the generation or scattering of axi-symmetric water waves by a system of permeable and impermeable co-axial cylinders. Various results concerning the generation and reflection of the axisymmetric surface or interfacial waves are derived in terms of Bessel functions. The resonance conditions within the trapped region are obtained in various cases. Further, expansions for multipole-line-source oblique-wave potentials are derived for both the cases of finite and infinite depth depending on the existence of the source point in a two-layered fluid.

Key words: porous barriers, reflection coefficient, source potentials, surface and interfacial waves, wave trapping

1. Introduction

The classical wavemaker theory developed by Havelock[1] has received significant attention in the literature under the assumption of the linearized theory of water waves in the case of a single homogeneous fluid of constant density for analyzing wave interaction with structures in two and three dimensions, including a large class of axi-symmetric problems. However, negligible progress has been made towards analyzing the wave motion of a two-layer or multi-layer fluid having different but constant densities with a free surface. This is of particular interest in understanding wave transformation in the presence of submerged/floating structures in continental shelves and estuaries. Such sharp density gradients can be generated in the ocean due to gravitational settling of sediments carried by fluids or by solar heating of the upper layer or in an estuary or a fjord into which fresh river water flows over oceanic water which is more saline and consequently heavier. This is being idealized by considering a two-layer system with lighter fluid of density ρ_1 lying over a heavier fluid of density ρ_2. Unlike the case of a homogeneous fluid, the two-layer fluid system, having a free surface and an interface, gives rise to fast modes (surface waves) as well as slow modes (internal waves). Hence, in the wave-scattering problem, the reflection and transmission characteristics of the structure will depend on both fast modes and slow modes. As a result, corresponding to each type of mode, the reflection and transmission coefficients, wave elevations, wave load on the submerged structures are to be analyzed, which needs a different type of treatment compared to that of wave motion in a single-layer fluid. In addition, the significant role of internal waves in a two-layer fluid is well understood from the various observations made in the literature [2].

It is observed that in the case of a two-layer fluid, ships experience an abnormal resistance force at the interface in the Norwegian fjords (see [3]) which was a mystery (and was attributed to dead water!) until Bjerknes, who explained it as due to the internal waves at the interface generated by the motion of the ship (see [4, pp. 234–242]). Yeung and Nguyen [5] employed an integral-equation technique to analyze the three-dimensional radiation and diffraction problems for a rectangular barge in finite depth. Cadby and Linton [6] used multipole expansions to solve problems involving submerged spheres in each of the two layers with the second layer being of infinite depth. Other notable works on wave–structure interaction in a two-layer fluid include [7–10].

One of the major difficulties in applying Havelock's wavemaker theory to a two layer system having a free surface is the existence of two surfaces where two types of waves propagate. In the fluid domain, apart from the conditions on the structure, flow discontinuity at the interface of the two layers makes the problem more difficult for applying Fourier analysis directly. As a result, the eigenfunctions involved in two-layer wave motion having a free surface are not orthogonal in the usual sense. Recently, Mandal and Chakrabarti [11] gave an integral-expansion formula, along with the corresponding inversion formula, for a function having an integrable singularity in $(0, \infty)$, in terms of orthogonal functions.

The scattering and generation of waves by permeable barriers are well studied in the recent literature [12–14] under the assumptions of the linearised theory of water waves in a single-layer fluid of finite depth. However, negligible progress has been made in the literature on the study of wave past porous structures in a two-layer fluid having a free surface and interface. Sherief *et al.* [15] analyzed the two-dimensional problem of forced gravity waves generated by a porous wavemaker in a two-layer fluid in water of finite and infinite depths in an elementary manner.

To deal with these classical problems in two-layer fluid systems with a free surface, in the present study, the velocity potentials for both the cases of finite and infinite depths are expanded in terms of a set of complete orthogonal eigenfunctions with respect to a suitable inner product. Utilizing these orthogonal functions, we obtain analytical solutions for the radiation and scattering of oblique water waves by a porous structure in in a two-layer fluid domain having a free surface. This form of solution is more transparent and straightforward compared to the one presented by Sherief *et al.* [15] for the particular case of normal incidence. Also, a class of problems dealing with the wave scattering and radiation of axi-symmetric water waves by permeable or(and) impermeable cylindrical structure(s) is(are) analyzed for a two-layer fluid medium. The results obtained for the single-fluid medium by Sahoo[16] concerning the generation and reflection of the surface waves are extended for a two-layer fluid medium and analytical expressions are obtained in terms of Bessel functions. In addition, the trapping of waves by the cylinders is studied and the condition for resonance of waves in the trapped region is derived. Numerical results concerning wave scattering and wave generation by permeable structures are analyzed for finite water depth for understanding of the derived theoretical results.

Further, we derive the oblique-line-source wave potentials in both the cases of water of finite and infinite depths depending on the existence of the source in the fluid domain and these are useful for the semi-analytical study of the scattering of water waves by submerged cylindrical structures of arbitrary shape in the fluid.

2. Mathematical formulation and solution method

We consider infinitesimal irrotational time-harmonic wave motion of two superimposed inviscid and incompressible fluids of constant density under the influence of gravity. We assume that the upper and lower fluid have constant densities ρ_1 and $\rho_2 (> \rho_1)$, respectively (see

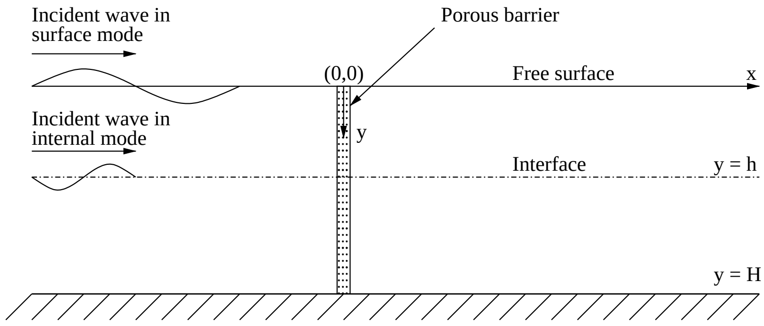

Figure 1. Defination sketch.

Figure 1). The mathematical formulation of the problems and their solution method for both the situations of finite and infinite depths are presented in the following subsections.

2.1. Oblique wave interaction with a vertical porous structure

Let (x, y, z) be the Cartesian co-ordinate of any point in the region with $o(0,0,0)$ being in the undisturbed free surface such that oy points vertically downwards. The fluid occupies the region $-\infty < x, z < \infty$, $0 < y < \infty$ for infinite depth and $-\infty < x, z < \infty$, $0 < y < H$ for finite depth, except the porous structure which is located at $x = 0$ and extends from the free surface to the bottom. In the case of finite depth, the upper layer is of depth h and the lower fluid is of depth $H - h$, whilst in case of infinite depth the lower fluid extends from h to ∞. For oblique-wave propagation with angular frequency ω, the velocity potential is assumed to be of the form $\Phi(x, y, z, t) = \Re\left\{\phi(x, y)e^{i(l_0 z - \omega t)}\right\}$, where $l_0 = \alpha \sin\nu$, $0 \le \nu \le \pi/2$ being the angle of the incident wave with x-axis and $\alpha = K(m_1)$ for infinite depth (finite depth) so that $l_0 \le \alpha$. The function $\phi(x, y)$ satisfies the partial differential equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - l_0^2\phi = 0, \tag{2.1}$$

along with the linearized free-surface boundary condition [17]

$$\frac{\partial\phi}{\partial y} + K\phi = 0 \text{ on } y = 0, \tag{2.2}$$

where $K = \dfrac{\omega^2}{g} > 0$, with g being the acceleration due to gravity, represents the normal incident wave number and the interface conditions [17]

$$\frac{\partial\phi(x,h-)}{\partial y} = \frac{\partial\phi(x,h+)}{\partial y}; \quad \rho_1\left[\frac{\partial\phi(x,h-)}{\partial y} + K\phi(x,h-)\right] = \rho_2\left[\frac{\partial\phi(x,h+)}{\partial y} + K\phi(x,h+)\right]. \tag{2.3}$$

When the porous wavemaker at $x = 0$ is assumed to oscillate with horizontal velocity of the form $\Re\left\{u(y)e^{i(l_0 z - wt)}\right\}$, the condition on the structure is given by

$$\frac{\partial\phi}{\partial x} = iG[\phi(0-, y) - \phi(0+, y)] + u(y), \text{ on } x = 0 \text{ for all } y \in (0,\infty) \text{ or } (0, H), \tag{2.4}$$

where $G = G_r + iG_i$ is the porous-effect parameter of dimension of length and defined by Yu and Chwang [18] as

$$G = \frac{\gamma(f + is)}{d(f^2 + s^2)}$$

with γ being the porosity constant, f the resistance force coefficient, s the inertial-force coefficient and d the thickness of the porous medium. The real part G_r represents the resistance effect of the porous material against the flow, while the imaginary part G_i represents the inertia effect of the fluid inside the porous material. We assume that the parameter G is the same in both layers of the fluid. Note that the limiting case $G \to 0$ represents the case of a rigid impermeable barrier. The detailed derivation of the porous-boundary condition is based on that of Sollitt and Cross [19] and is well explained in [18]. It is widely used in the recent literature and was recently reviewed by Chwang and Chan [20]. This formula is based on the principle that the horizontal velocity is proportional to the pressure gradient; a simple version of this porous-boundary condition was derived earlier by Chwang [12].

Finally, the radiation conditions for the outgoing waves are given by

$$\phi(x,y) \sim \begin{cases} A_1 e^{i\mu x - Ky} + R_1 e^{-i\mu x - Ky} + A_2 e^{i\beta x} l(y) + R_2 e^{-i\beta x} l(y) \text{ as } x \to -\infty \\ T_1 e^{i\mu x - Ky} + T_2 e^{i\beta x} l(y) \text{ as } x \to \infty, \end{cases} \tag{2.5}$$

for infinite depth and

$$\phi(x,y) \sim \begin{cases} \sum_{j=1}^{2} A_j e^{i\mu_j x} f_0(m_j, y) + \sum_{j=1}^{2} R_j e^{-i\mu_j x} f_0(m_j, y) \text{ as } x \to -\infty \\ \sum_{j=1}^{2} T_j e^{i\mu_j x} f_0(m_j, y), \text{ as } x \to \infty \end{cases} \tag{2.6}$$

for finite depth, where the constants $R_j, T_j, j = 1, 2$ are unknown constants to be determined while the constants $A_j, j = 1, 2$ are assumed to be known and $\mu = \sqrt{K^2 - l_0^2}$, $\beta = \sqrt{v^2 - l_0^2}$, $\mu_j = \sqrt{m_j^2 - l_0^2}$, $j = 1, 2$.

For infinite depth, the bottom-boundary condition is given by

$$\phi, \nabla\phi \to 0 \text{ as } y \to \infty \tag{2.7}$$

and is replaced by

$$\frac{\partial \phi}{\partial y} = 0 \text{ on } y = H \tag{2.8}$$

in the case of finite depth.

The above boundary-value problem for ϕ represents a class of problems concerning the generation or scattering of oblique water waves by a vertical porous structure. The complete analytical solution of this general problem is obtained and, then, particular choices of the constants $A_i, R_j, j = 1, 2$ and the function $u(y)$ illustrate specific problems of physical interest.

2.1.1. *Case of infinite depth*

The general form of the velocity potential $\phi(x,y), 0 < y < \infty$, satisfying Equation (2.1) and the boundary conditions (2.2–2.5) and (2.7), is given by

$$\phi(x,y) = \begin{cases} A_1 e^{i\mu x - Ky} + R_1 e^{-i\mu x - Ky} + A_2 e^{i\beta x} l(y) + R_2 e^{-i\beta x} l(y) \\ \qquad + \int_0^\infty A(\xi) L(\xi, y) e^{x\sqrt{\xi^2 + l_0^2}} d\xi, \quad x < 0, \\ T_1 e^{i\mu x - Ky} + T_2 e^{i\beta x} l(y) + \int_0^\infty B(\xi) L(\xi, y) e^{-x\sqrt{\xi^2 + l_0^2}} d\xi, \quad x > 0, \end{cases}$$

where $A(\xi), B(\xi)$ are unknown functions to be determined and the bounded functions $e^{-Ky}, 0<y<\infty,\ L(\xi,y)=\begin{cases} L_1(\xi,y), & \text{for } y\in(0,h)\\ L_2(\xi,y), & \text{for } y\in(h,\infty),\end{cases}$ with

$$L_1(\xi,y)=K(\xi\cos\xi y-K\sin\xi y),$$

$$L_2(\xi,y)=L_1(\xi,y)+\frac{(\rho_2-\rho_1)}{\rho_2}(\xi^2+K^2)\sin\xi h\cos\xi(y-h),$$

and

$$l(y)=\begin{cases} g(y), & \text{for } y\in(0,h)\\ e^{v(h-y)}, & \text{for } y\in(h,\infty),\end{cases}$$

with

$$g(y)=\frac{K(\rho_2+\rho_1)-v(\rho_2-\rho_1)}{2K\rho_1}e^{-v(y-h)}+\frac{(K-v)(\rho_2-\rho_1)}{2K\rho_1}e^{v(y-h)}.$$

Here v is the unique positive real root of the transcendental equation

$$(K+v)(\rho_2-\rho_1)e^{-vh}+[K(\rho_2+\rho_1)-v(\rho_2-\rho_1)]e^{vh}=0. \tag{2.9}$$

The functions $e^{-Ky}, L(\xi,y)$ and $l(y)$ are orthogonal with respect to the inner product defined by

$$<\phi_1,\psi_1>_1=\lim_{\epsilon\to 0}\left[\rho_1\int_0^h e^{-\epsilon y}\phi_1(y)\psi_1(y)dy+\rho_2\int_h^\infty e^{-\epsilon y}\phi_1(y)\psi_1(y)dy\right], \tag{2.10}$$

where ϕ_1 and ψ_1 belong to the real Hilbert space of integrable functions. These functions are the eigenfunctions of the self-adjoint linear differential operator $\dfrac{d^2}{dy^2}$ corresponding to the eigenvalues K^2, v^2 and $-\xi^2(\xi>0)$, satisfying the boundary conditions (2.2), (2.3) and (2.7). Hence they form a complete set.

In the above expansion for the velocity potential, K refers to the wave number corresponding to wave propagating on the free surface (surface mode and is referred as SM), whilst v refers to the wave number corresponding to the propagating wave generated due to the presence of the interface (internal mode and is referred as IM). In addition, it is worth mentioning that, by allowing $\rho_1\to\rho_2$ and $h\to 0$ simultaneously and noting that Equation (2.9) does not have any positive real root, we obtain Havelock's type of expansion for the velocity potential in a single layer fluid.

The above orthogonal relation is utilized to determine the unknowns in the expansion for the function $\phi(x,y)$ and is described as follows.

Using the continuity of horizontal velocity $\partial\phi/\partial x$ across $x=0$, and the orthogonal property of the functions involved, we derive that $A_j-R_j=T_j, j=1,2;\quad A(\xi)=-B(\xi)$. Further, utilizing the orthogonal relation on the porous-wavemaker condition (2.4), we obtain that

$$i\mu(A_1-R_1)-2iR_1G=\frac{2Ke^{2Kh}\left\langle u(y),e^{-Ky}\right\rangle_1}{\rho_2+\rho_1\left(e^{2Kh}-1\right)},$$

$$i\beta(A_2-R_2)-2iR_2G=\frac{2v}{\rho_2+2\rho_1 v\int_0^h g^2(y)dy}\left[\rho_1\int_0^h u(y)g(y)dy+\rho_2\int_h^\infty u(y)e^{-v(y-h)}dy\right]$$

and

$$A(\xi)=\frac{2}{\pi}\frac{\rho_2}{(\xi^2+K^2)\Big[\sqrt{\xi^2+l_0^2}-2\mathrm{i}G\Big]\mathcal{D}_0(\xi)}\left[\rho_1\int_0^h u(y)L_1(\xi,y)\mathrm{d}y+\rho_2\int_h^\infty u(y)L_2(\xi,y)\mathrm{d}y\right], \tag{2.11}$$

where

$$\mathcal{D}_0(\xi)=\Big[(\rho_2-\rho_1)\xi\sin\xi h+K\rho_2\cos\xi h\Big]^2+\rho_1^2K^2\sin^2\xi h.$$

Thus, the unknowns in the expansion for the velocity potential in the two-layer fluid are obtained in a straightforward manner. Next, as an application of this expansion, we analyse briefly the porous-wavemaker problem and the wave scattering by a porous barrier in a two-layer fluid of infinite depth.
Wavemaker problem: In this case $A_1=0, A_2=0$ and $u(y)\neq 0$ in the general expansion for the velocity potential, as there are no incident waves. The unknown complex constants R_1, R_2 represent the far-field wave amplitudes of the surface and interfacial waves generated by the porous wavemaker and are obtained as

$$R_1=\frac{2\mathrm{i}K\mathrm{e}^{2Kh}\Big\langle u(y),\mathrm{e}^{-Ky}\Big\rangle_1}{(\mu+2G)[\rho_2+\rho_1(\mathrm{e}^{2Kh}-1)]},$$

$$R_2=\frac{2\mathrm{i}v}{(\beta+2G)\Big[\rho_2+2\rho_1 v\int_0^h g^2(y)\mathrm{d}y\Big]}\left[\rho_1\int_0^h u(y)g(y)\mathrm{d}y+\rho_2\int_h^\infty u(y)\mathrm{e}^{-v(y-h)}\mathrm{d}y\right],$$

where $A(\xi)$ remains as given by the relation (2.11).

In particular, when $u(y)=\mathrm{e}^{-Ky}$, utilizing the symmetric potential $\phi(x,y)$, we obtain the wave amplitudes R_1 and R_2, the force acting on the wavemaker F and the surface and interfacial elevations $\zeta_1(x,t)$ and $\zeta_2(x,t)$ as

$$R_1=\frac{\mathrm{i}}{\mu+2G},\quad R_2=\frac{\mathrm{i}}{\beta+2G},$$

$$\begin{aligned}F(z)&=\mathrm{i}\omega\mathrm{e}^{\mathrm{i}l_0z}\left[\rho_1\int_0^h(\phi(0+,y)-\phi(0-,y))\mathrm{d}y+\rho_2\int_h^\infty(\phi(0+,y)-\phi(0-,y))\mathrm{d}y\right]\\&=\frac{2\omega[\rho_1+(\rho_2-\rho_1)\mathrm{e}^{-Kh}]}{K(\mu+2G)}\mathrm{e}^{\mathrm{i}l_0z},\end{aligned}$$

$$\zeta_1(\pm x,z,t)=-\mathfrak{Re}\left\{\frac{\mathrm{i}\omega}{g}\phi(x,0)\mathrm{e}^{\mathrm{i}(l_0z-\omega t)}\right\}=\mp\frac{\omega}{g}\mathfrak{Re}\left\{\frac{\pm\mathrm{e}^{\mathrm{i}(\mu x+l_0z-\omega t)}}{\mu+2G}\right\},\quad x>0$$

and

$$\begin{aligned}\zeta_2(\pm x,z,t)&=-\mathfrak{Re}\left\{\frac{\mathrm{i}\omega}{g(\rho_2-\rho_1)}\Big[\rho_2\phi(x,h+)-\rho_1\phi(x,h-)\Big]\mathrm{e}^{\mathrm{i}(l_0z-\omega t)}\right\}\\&=\mp\frac{\omega\mathrm{e}^{-Kh}}{g}\mathfrak{Re}\left\{\frac{\pm\mathrm{e}^{\mathrm{i}(\mu x+l_0z-\omega t)}}{\mu+2G}\right\},\quad x>0.\end{aligned}$$

Oblique wave scattering: In the case of the scattering problem, the permeable barrier is kept fixed at $x=0$ and hence $u(y)=0$. In addition, in this case $A_1=1$ (or) $0, A_2=0$ (or) 1 because of the presence of one of the incident waves, as appropriate, and the unknown constants R_1, T_1 (or) R_2, T_2, representing the reflection and transmission coefficients associated with the free surface or interfacial wave, are given by

$$R_1 = \frac{\mu}{\mu + 2G} \quad \text{(or)} \quad R_2 = \frac{\beta}{\beta + 2G}$$

and

$$T_1 = \frac{2G}{\mu + 2G} \quad \text{(or)} \quad T_2 = \frac{2G}{\beta + 2G}.$$

From the above relations, it is observed that, in either case, energy loss occurs at the porous barrier,*i.e.*, $|R_j|^2 + |T_j|^2 < 1, j = 1, 2$. Also, when $G \to 0$, $R_j = 1, T_j = 0, j = 1, 2$ showing no energy loss in the case of a rigid barrier.

2.1.2. *Case of finite depth*

In this case, the general form of the velocity potential $\phi(x, y)$, $0 < y < H$ satisfying equation (2.1) and the boundary conditions (2.2–2.4), (2.6) and (2.8) is given by

$$\phi(x,y) = \begin{cases} \sum_{j=1}^{2} A_j e^{i\mu_j x} f_0(m_j, y) + \sum_{j=1}^{2} R_j e^{-i\mu_j x} f_0(m_j, y) + \sum_{n=1}^{\infty} B_n f_n(p_n, y) e^{x\sqrt{p_n^2 + l_0^2}}, & x < 0, \\ \sum_{j=1}^{2} T_j e^{i\mu_j x} f_0(m_j, y) + \sum_{n=1}^{\infty} C_n f_n(p_n, y) e^{-x\sqrt{p_n^2 + l_0^2}}, & x > 0, \end{cases}$$

where B_n, C_n, for $n = 1, 2, 3\ldots$ are unknown constants to be determined and the functions

$$f_0(m_j, y) = \begin{cases} \sinh m_j(h - H)[m_j \cosh m_j y - K \sinh m_j y] & \text{for } y \in (0, h) \\ [m_j \sinh m_j h - K \cosh m_j h] \cosh m_j (y - H) & \text{for } y \in (h, H), \end{cases} \quad j = 1, 2, \tag{2.12}$$

and

$$f_n(p_n, y) = \begin{cases} \sin p_n(h - H)[p_n \cos p_n y - K \sin p_n y] & \text{for } y \in (0, h) \\ [p_n \sin p_n h + K \cos p_n h] \cos p_n (y - H) & \text{for } y \in (h, H), \end{cases} \quad n = 1, 2, 3\ldots, \tag{2.13}$$

with $0 < m_1 < m_2$ (say). Here $ip_n, n = 1, 2, 3, \ldots, p_n > 0$ are the roots of the dispersion relation

$$(\rho_2 - \rho_1)x^2 - \rho_2 K\Big[\coth x(H - h) + \coth xh\Big]x + K^2\Big[\rho_1 + \rho_2 \coth x(H - h) \coth xh\Big] = 0.$$

The bounded functions f_n's are orthogonal with respect to the inner product

$$< f_n, f_m >_2 = \rho_1 \int_0^h f_n(y) f_m(y) \mathrm{d}y + \rho_2 \int_h^H f_n(y) f_m(y) \mathrm{d}y. \tag{2.14}$$

These functions are the eigenfunctions of the self-adjoint linear differential operator $\frac{\mathrm{d}^2}{\mathrm{d}y^2}$ corresponding to the eigenvalues m_j^2, $j = 1, 2$ and $-p_n^2, n = 1, 2, 3, \ldots$ and satisfying the boundary conditions (2.2), (2.3) and (2.8). Hence they form a complete set.

In this case also, there are two types of progressive wave, which are generated because of the presence of the free surface and the interface for the wave motion in a two-layer fluid. The eigenvalues m_1 and m_2 correspond to the wave numbers of the incident waves in SM and IM respectively, whilst the p_n correspond to the evanescent wave modes. It may be noted that by allowing $\rho_1 \to \rho_2$ and $h \to 0$ simultaneously, the above eigenfunctions reduce to those used in water-wave problems for a single-layer fluid having a free surface. We will now use of the above orthogonal relation to derive unknown constants in the expansion for the velocity potential.

Utilizing the continuity of velocity across the porous wavemaker, *i.e.*, the continuity of $\partial\phi/\partial x$ across $x=0$ and the orthogonal property of the functions $f_0(m_j, y), j=1,2$ and $f_n(p_n, y), n=1,2,3\ldots$, we derive that $A_j - R_j = T_j, j=1,2;\quad B_n = -C_n,\quad n=1,2,3\ldots$.

Using the condition (2.4) on the porous wavemaker, and the orthogonality relation as mentioned above, the unknown constants $R_j, j=1,2, B_n, n=1,2,3\ldots$ are obtained as

$$R_j = \frac{\mu_j}{\mu_j+2G}A_j + i\frac{\left\langle u(y), f_0(m_j, y)\right\rangle_2}{M_0(m_j)(\mu_j+2G)},\quad j=1,2,\quad B_n = \frac{\left\langle u(y), f_n(p_n, y)\right\rangle_2}{M_n(p_n)\left[\sqrt{p_n^2+l_0^2}-2iG\right]},\quad n=1,2,3\ldots, \tag{2.15}$$

where $M_0(m_j) = \left\langle f_0(m_i, y), f_0(m_i, y)\right\rangle_2, j=1,2$ and $M_n(p_n) = \left\langle f_n(p_n, y), f_n(p_n, y)\right\rangle_2, n=1,2,3\ldots$. Next, we deduce from the general expansion formula the results for the generation of water waves by a porous wavemaker and for the scattering of waves by a porous barrier in a two-layer fluid of finite depth.

Wavemaker Problem: Similar to the case of water of infinite depth, in this case also $A_1=0$, $A_2=0$ and $u(y)\neq 0$. The unknown constants R_1, R_2 represent the far-field wave amplitudes of the surface and interfacial waves generated by the porous wavemaker located at $x=0$ and are obtained as

$$R_j = \frac{\mathrm{i}\left\langle u(y), f_0(m_j, y)\right\rangle_2}{(\mu_j+2G)M_0(m_j)},\quad j=1,2.$$

The constants $B_n's$ remain the same as in the relation (2.15).

In particular, when $u(y)=1$ for $0<y<H$, the wave amplitudes R_1 and R_2, the force F acting on the wavemaker and the surface and interfacial elevations $\zeta_1(x,t)$ and $\zeta_2(x,t)$ are given by

$$\begin{aligned} R_j = {} & \frac{\mathrm{i}\rho_1 \sinh m_j(h-H)[m_j\sinh m_j h + K(1-\cosh m_j h)]}{m_j(\mu_j+2G)M_0(m_j)} \\ & - \frac{\mathrm{i}\rho_2[m_j\sinh m_j h - K\cosh m_j h]\sinh m_j(h-H)}{m_j(\mu_j+2G)M_0(m_j)},\quad j=1,2, \end{aligned} \tag{2.16}$$

$$F(z) = -2\omega e^{\mathrm{i}l_0 z}\sum_{j=1}^{2} R_j^2(m_j+2G)M_0(m_j) - 2\mathrm{i}\omega e^{\mathrm{i}l_0 z}\sum_{n=1}^{\infty} B_n^2\left(\sqrt{p_n^2+l_0^2}-2\mathrm{i}G\right)M_n(p_n),$$

$$\begin{aligned} \zeta_1(\pm x, z, t) = \pm\frac{\omega}{g}\Re\mathfrak{e}\Bigg\{ & \sum_{j=1}^{2}\mathrm{i}R_j m_j \sinh m_j(h-H)\ e^{\mathrm{i}(\pm\mu_j x + l_0 z - \omega t)} \\ & + \sum_{n=1}^{\infty}\mathrm{i}B_n p_n \sin p_n(h-H)\ e^{\left(\mp\sqrt{p_n^2+l_0^2}x + \mathrm{i}l_0 z - i\omega t\right)}\Bigg\},\quad x>0 \end{aligned}$$

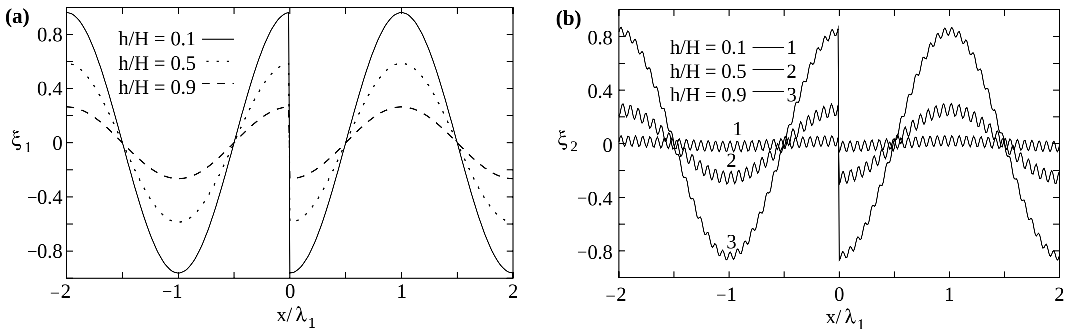

Figure 2. (a) Free surface and (b) Interface elevation vs. x/λ_1 for different h/H values with $s=0.9$, $G=1+0.5i$ and $\nu=\pi/3.0$.

and

$$\zeta_2(\pm x, z, t) = \pm\frac{\omega}{g(\rho_2-\rho_1)}\Re\mathfrak{e}\left\{\sum_{j=1}^{2} i R_j \mathrm{e}^{\mathrm{i}(\pm\mu_j x + l_0 z - \omega t)} Q_0(m_j) + \sum_{n=1}^{\infty} \mathrm{i} B_n \mathrm{e}^{\left(\mp\sqrt{p_n^2+l_0^2}x + \mathrm{i}l_0 z - \mathrm{i}\omega t\right)} Q_n(p_n)\right\}, \quad x>0.$$

Here

$$\begin{aligned} Q_0(m_j) &= \rho_2 \cosh m_j(h-H)[m_j \sinh m_j h - K\cosh m_j h] \\ &\quad -\rho_1 \sinh m_j(h-H)[m_j \cosh m_j h - K \sinh m_j h], \quad j=1,2, \\ Q_n(p_n) &= \rho_2 \cos p_n(h-H)[p_n \sin p_n h + K\cos p_n h] \\ &\quad -\rho_1 \sin p_n(h-H)[p_n \cos p_n h - K\sin p_n h], \quad n=1,2,3,\ldots, \end{aligned}$$

$$\begin{aligned} B_n &= \frac{\rho_1 \sin p_n(h-H)[p_n \sin p_n h - K(1-\cos p_n h)]}{p_n\left(\sqrt{p_n^2+l_0^2}-2\mathrm{i}G\right)M_n(p_n)} \\ &\quad -\frac{\rho_2[p_n \sin p_n h + K\cos p_n h]\sin p_n(h-H)}{p_n\left(\sqrt{p_n^2+l_0^2}-2\mathrm{i}G\right)M_n(p_n)}, \quad n=1,2,3,\ldots, \end{aligned} \tag{2.17}$$

and R_j, $j=1,2$ are given by the relation (2.16).

Using the above expressions, we now compute and analyze the surface and interfacial elevations and force on the structure at $z=0$ for the case of waves making an angle $\pi/6$ with x-axis.

In Figure 2a, b, the free-surface and interface elevations are plotted, respectively, for different h/H ratios. It is observed that, with an increase in h/H ratio, the wave amplitude at the interface is increasing in nature, whilst the wave amplitude at the free surface is decreasing. At the interface the surface elevation consists of two wave patterns, namely the primary and secondary wave pattern. The secondary wave pattern is observed to be reducing for higher values of h/H.

Variation of free surface, interface elevations at different s values are plotted in Figure 3a, b, respectively. It is observed that when the density difference is large, the amplitude of both free surface and interface are higher. At the interface, for higher values of the s too, the surface elevation consists of the primary and secondary wave patterns and the former is

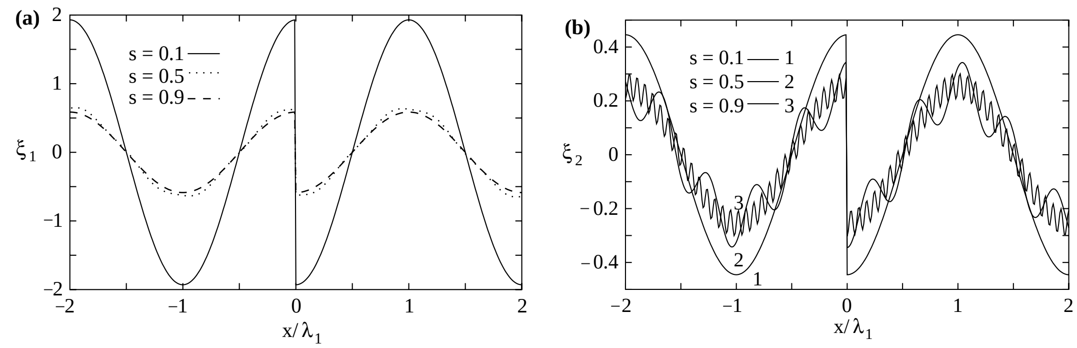

Figure 3. (a) Free surface and (b) Interface elevation versus x/λ_1 for different s values with $h/H=0.5$, $G=1+0.5i$ and $\nu=\pi/3.0$.

similar to that of the free-surface wave pattern with the latter being sensitive to higher values of s. Also, the amplitude of the interface elevation is larger compared to that of the free surface and could cause more resistance to the structure.

Based on the expression for the force F acting on the wavemaker, in Figure 4, the non-dimensional force amplitude K_f (defined as $K_f=|K^2F/2\omega\rho_1 u|$) against m_1H for different s values is plotted. It is observed that the amplitude of the force acting on the structure increases as m_1H and s increases. This may be due to the existence of the the free surface and interface which produces two types of wave pattern existing at the interface as discussed in Figure 3a and b.

Wave scattering: Here $A_1=1$ (or) $0, A_2=0$ (or) 1 and $u(y)=0$. The constants R_1, T_1 (or) R_2, T_2 represent the reflection and transmission coefficients of the surface or interfacial incidental wave and are given by

$$R_1=\frac{\mu_1}{\mu_1+2G} \quad \text{or} \quad R_2=\frac{\mu_2}{\mu_2+2G}$$

and

$$T_1=\frac{2G}{\mu_1+2G} \quad \text{or} \quad T_2=\frac{2G}{\mu_2+2G}.$$

Clearly, there is an energy loss due to the porous barrier, *i.e.*, $|R_j|^2+|T_j|^2<1$, $j=1,2$. When $G=0$, the incident wave is fully reflected with no loss of energy by the barrier.

The reflection and transmission coefficients in surface and interfacial modes depend on the respective wave numbers, the porous-effect parameter and the angle of the incidence wave. The variation of the reflection coefficients (both in surface and internal modes) with respect to m_1H for different s-values are presented in Figure 5. It is observed that with an increase in m_1H the reflection coefficients, both in surface and internal modes, decrease from its peak sharply and then maintains a constant value. The reflection coefficient in the surface mode is high for smaller values of s and an opposite trend is observed for the internal mode. However, over the entire range of m_1H the reflection coefficient in the internal mode is found to be higher than that in the surface mode.

In Figure 6, the variation of the reflection coefficients with respect to m_1H for different h/H values are shown. It is observed that the effect of interface location on the reflection coefficient in the surface mode is negligible. However a negligible variation is noted for the internal modes in the shallow-water region.

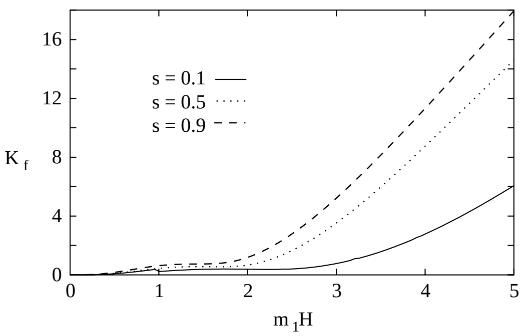

Figure 4. Non-dimensional force vs. m_1H for different s values with $h/H = 0.5$, $G = 1 + 0.5i$ and $\nu = \pi/3.0$.

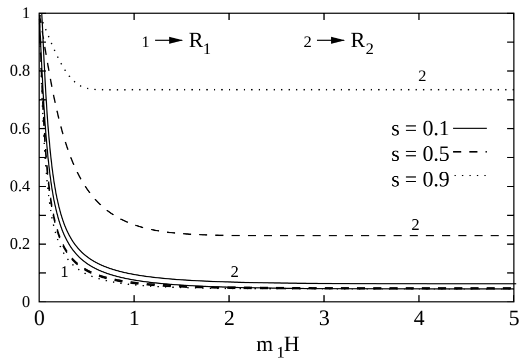

Figure 5. Reflection coefficients vs. m_1H for different s values with $h/H = 0.5$, $G = 1 + 0.5i$ and $\nu = \pi/3.0$.

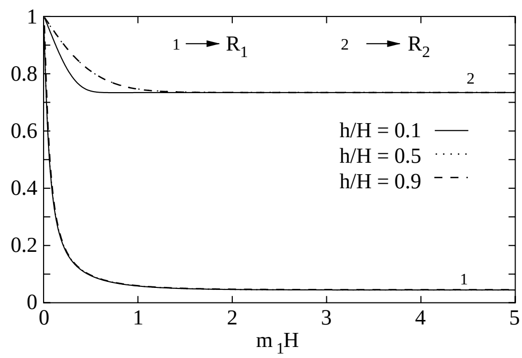

Figure 6. Reflection coefficients vs. m_1H for different h/H ratios with $G = 1 + 0.5i$, $s = 0.9$ and $\nu = \pi/3.0$.

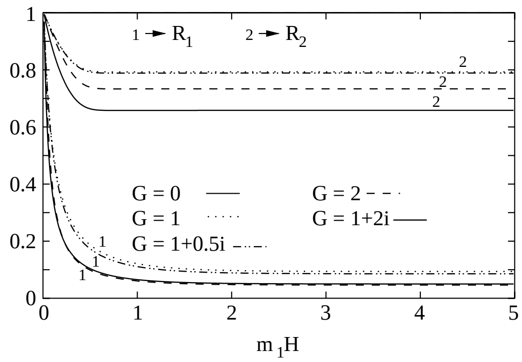

Figure 7. Reflection coefficients vs. m_1H for different G values with $h/H = 0.5$, $s = 0.9$ and $\nu = \pi/3.0$.

Finally, the variation of reflection coefficients against m_1H for different G values is plotted in Figure 7. It is observed that the reflection coefficients in both surface and internal modes are decreasing with an increase in the porous-effect parameter. In general, for a porous structure, reflection of waves in IM is higher than that of waves in SM.

2.2. Generation of axi-symmetric waves

In this section, we undertake the study of axi-symmetric three-dimensional water waves in a two-layer fluid having different but constant densities for both the cases of infinite and finite depth. Let (r, θ, y) be any point in the cylindrical coordinate system with $o(0, 0, 0)$ being the origin in the undisturbed free surface such that oy points vertically downwards. We first demonstrate the solution method to the general boundary-value problem comprising the generation and scattering of axi-symmetric water waves in a two-layer fluid, either by two co-axial cylinders of impermeable, permeable types placed at $r = a$, b, respectively, with $0 < a < b$ or by a single permeable cylinder placed at $r = b$. Subsequently, we discuss the specific cases of physical interest for both infinite and finite depths. However, numerical results are presented for the case of finite depth in order to understand the phenomenon of trapping and wave resonance in SM and IM.

Under the assumption of the linearised theory of water waves, the axi-symmetric water wave motion is described by the velocity potential $\Phi(r, y, t) = \Re\mathfrak{e}\{\phi(r, y)\mathrm{e}^{-\mathrm{i}\omega t}\}$. Then the function $\phi(r, y)$ satisfies the partial differential equation

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial y^2}=0, \quad \text{in the fluid region,} \tag{2.18}$$

along with the free-surface condition (2.2), the interface conditions (2.3) and the bottom condition (2.7) or (2.8) depending upon the case of water of infinite or finite depth.

Further, the conditions on the inner impermeable cylinder and the outer permeable cylinder are given by

$$\frac{\partial\phi(a+, y)}{\partial r}=u_1(y), a>0 \tag{2.19}$$

and

$$\frac{\partial\phi(b\pm, y)}{\partial r}=\mathrm{i}G(\phi(b-, y)-\phi(b+, y))+u_2(y), b>a, \tag{2.20}$$

where $u_1(y)$, $u_2(y)$ represent the velocities of the oscillating inner and outer cylinders, respectively, and G is the porous-effect parameter as mentioned in Subsection 2.1, which is assumed to be the same in both layers of the fluid.

The radiation condition, in the case of water of infinite depth, is given by

$$\phi(r, y)\sim A_1H_0^{(2)}(Kr)\mathrm{e}^{-Ky}+R_1H_0^{(1)}(Kr)\mathrm{e}^{-Ky}+A_2H_0^{(2)}(vr)l(y)+R_2H_0^{(1)}(vr)l(y), \tag{2.21}$$

as $(b<)r\to\infty$, where R_j, $j=1,2$ are the unknown constants to be determined. Here $H_n^{(m)}(x)$ for $m=1,2$ are the Hankel functions of order n and the constants A_j, $j=1,2$ are assumed to be known.

$$\phi(r, y)\sim\sum_{j=1}^{2}\Big[A_jH_0^{(2)}(m_jr)+R_jH_0^{(1)}(m_jr)\Big]f_0(m_j, y) \text{ as } (b<)r\to\infty. \tag{2.22}$$

The detailed solution procedures are discussed separately for both the cases of water of infinite and finite depths and physical problems of practical interest are analyzed as special cases of the general problem.

2.2.1. *Case of infinite depth with co-axial cylinders*

The general form of the velocity potential ϕ which satisfies the Laplace equation as given by the boundary-value problem described by the relations (2.2), (2.3), (2.7), (2.18–2.21) is given by

$$\phi(r, y)=\begin{cases} \Big[\alpha_1J_0(Kr)+\beta_1H_0^{(1)}(Kr)\Big]\mathrm{e}^{-Ky}+\Big[\alpha_2J_0(vr)+\beta_2H_0^{(1)}(vr)\Big]l(y) \\ +\displaystyle\int_0^\infty\Big[A(\xi)I_0(\xi r)+B(\xi)K_0(\xi r)\Big]L(\xi, y)\mathrm{d}\xi, \quad 0<a<r<b, \quad 0<y<\infty, \\ A_1H_0^{(2)}(Kr)\mathrm{e}^{-Ky}+R_1H_0^{(1)}(Kr)\mathrm{e}^{-Ky}+A_2H_0^{(2)}(vr)l(y)+R_2H_0^{(1)}(vr)l(y) \\ +\displaystyle\int_0^\infty C(\xi)K_0(\xi r)L(\xi, y)\mathrm{d}\xi, \quad b<r<\infty, \quad 0<y<\infty, \end{cases} \tag{2.23}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2, R_1, R_2, A(\xi), B(\xi)$ and $C(\xi)$ are unknowns to be determined. Also, the functions $J_n(x)$ for $n=0,1$ are Bessel functions of the first kind, and $I_n(x), K_n(x)$ for $n=0,1$ are modified Bessel functions of the first and second kind, respectively.

Here, for notational convenience, we define

$$a_j(K)=\left\langle u_j,\mathrm{e}^{-Ky}\right\rangle_1=\left[\rho_1\int_0^h u_j(y)\mathrm{e}^{-Ky}\mathrm{d}y+\rho_2\int_h^\infty u_j(y)\mathrm{e}^{-Ky}\mathrm{d}y\right],$$
$$b_j(v)=\left\langle u_j,l(y)\right\rangle_1=\left[\rho_1\int_0^h u_j(y)l(y)dy+\rho_2\int_h^\infty u_j(y)l(y)\mathrm{d}y\right],$$
$$d_j(\xi)=\left\langle u_j,L(\xi,y)\right\rangle_1=\left[\rho_1\int_0^h u_j(y)L_1(\xi,y)\mathrm{d}y+\rho_2\int_h^\infty u_j(y)L_2(\xi,y)\mathrm{d}y\right],\quad j=1,2.$$

It is now a routine matter to determine the unknown constants. We explain briefly how this can be done. Using the continuity of the velocity across the porous wall at $r=b$, the condition (2.20) for the porous wall at $r=b$ and the condition (2.19) on the inner cylinder at $r=a$, we derive three equations in terms of the orthogonal functions discussed in Section 2.1.1. Applying the orthogonal property and after some algebraic calculations, we can determine all the known constants and unknown functions in terms of Bessel functions.

We directly express the unknown constants for specific problems of physical interest in the foregoing analysis. We remark here that, to understand the physically interesting phenomenon, it is sufficient just to look at the unknown constants in the relation (2.25).

Waves generated by the inner cylinder while the outer cylinder is kept fixed. In this case $u_1(y)\neq 0, u_2(y)=0, A_1=0$ and $A_2=0$. This is the situation of physical interest where wave resonance takes place for certain wave numbers. With the notations

$$\Delta_2(x)=J_1(xb)H_1^{(1)}(xa)-J_1(xa)H_1^{(1)}(xb) \tag{2.24}$$

and

$$\Delta_3(x)=2GH_1^{(1)}(xa)+\pi bx^2\Delta_2(x)H_1^{(1)}(xb) \tag{2.25}$$

and the constraints that Δ_2,Δ_3 never vanishes at the wave numbers, the unknown constants are determined as

$$R_1=\frac{4G\mathrm{e}^{2Kh}a_1(K)}{\pi Kb[\rho_2+\rho_1(\mathrm{e}^{2Kh}-1)]\Delta_3(K)}$$

$$\alpha_2=R_2\frac{H_1^{(1)}(va)H_1^{(1)}(vb)}{\Delta_2(v)}-\frac{H_1^{(1)}(vb)}{\Delta_2(v)}\frac{2b_1(v)}{[\rho_2+2v\rho_1\int_0^h g^2(y)\mathrm{d}y]},$$

$$\beta_2=-R_2\frac{J_1(va)H_1^{(1)}(vb)}{\Delta_2(v)}+\frac{J_1(vb)}{\Delta_2(v)}\frac{2b_1(v)}{[\rho_2+\rho_1\int_0^h g^2(y)\mathrm{d}y]},$$

$$R_2=\frac{4Gb_1(v)}{\pi vb[\rho_2+2v\rho_1\int_0^h g^2(y)\mathrm{d}y]\Delta_3(v)},$$

Note that when $\Delta_2(K)=0$ or $\Delta_2(v)=0$, the phenomenon of resonance occurs between the cylinders.

Waves generated by the outer cylinder while the inner cylinder is kept fixed. Physically this is an interesting case in which wave trapping occurs, a phenomenon observed when a wave propagates into a region bounded by two walls. Here we must choose $u_1(y)=0, u_2(y)\neq 0, A_1=0$

and $A_2=0$. Then, the unknowns, under the constraints $\Delta_2(K)\neq 0$ and $\Delta_2(v)\neq 0$, are determined as

$$\alpha_1=R_1\frac{H_1^{(1)}(Ka)H_1^{(1)}(Kb)}{\Delta_2(K)},\quad \beta_1=-R_1\frac{J_1(Ka)H_1^{(1)}(Kb)}{\Delta_2(K)},\quad R_1=-\frac{2Ka_2(K)\Delta_2(K)\mathrm{e}^{2Kh}}{[\rho_2+\rho_1(\mathrm{e}^{2Kh}-1)]\Delta_3(K)}$$

and

$$\alpha_2=R_2\frac{H_1^{(1)}(va)H_1^{(1)}(vb)}{\Delta_2(v)},\quad \beta_2=-R_2\frac{J_1(va)H_1^{(1)}(vb)}{\Delta_2(v)},\quad R_2=-\frac{2vb_2(v)\Delta_2(v)}{[\rho_2+2v\rho_1\int_0^h g^2(y)\mathrm{d}y]\Delta_3(v)}.$$

Clearly, it may be seen that in the cases of either $\Delta_2(K)=0$ or $\Delta_2(v)=0$, the generated wave gets trapped between the cylinders while it maintains a tranquil zone outside the cylinders.

Scattering of waves when the co-axial cylinders are kept fixed. This is the scattering case with resonance. For surface-wave mode incidence, we must choose $u_1(y)=0, u_2(y)=0, A_1=1(0)$ and $A_2=0(1)$. The unknowns, under the constraint that $\Delta_2(K)\neq 0$, are determined as

$$\alpha_1=R_1\frac{H_1^{(1)}(Ka)H_1^{(1)}(Kb)}{\Delta_2(K)}+\frac{H_1^{(1)}(Ka)H_1^{(2)}(Kb)}{\Delta_2(K)},$$
$$\beta_1=-R_1\frac{J_1(Ka)H_1^{(1)}(Kb)}{\Delta_2(K)}-\frac{J_1(Ka)H_1^{(2)}(Kb)}{\Delta_2(K)},$$

$$R_1=-\frac{\pi bK^2\Delta_2(K)H_1^{(2)}(Kb)+2GH_1^{(2)}(Ka)}{\pi bK^2\Delta_2(K)H_1^{(1)}(Kb)+2GH_1^{(1)}(Ka)},$$

and $\alpha_2=\beta_2=R_2=A(\xi)=B(\xi)=C(\xi)=0$.

For the interfacial wave incidence, the unknown, under the constraint that $\Delta_2(v)\neq 0$, are determined as

$$\alpha_2=R_2\frac{H_1^{(1)}(va)H_1^{(1)}(vb)}{\Delta_2(v)}+\frac{H_1^{(1)}(va)H_1^{(2)}(vb)}{\Delta_2(v)},$$
$$\beta_2=-R_2\frac{J_1(va)H_1^{(1)}(vb)}{\Delta_2(v)}-\frac{J_1(va)H_1^{(2)}(vb)}{\Delta_2(v)},$$
$$R_2=-\frac{\pi bv^2\Delta_2(v)H_1^{(2)}(vb)+2GH_1^{(2)}(va)}{\pi bv^2\Delta_2(v)H_1^{(1)}(vb)+2GH_1^{(1)}(va)},$$

and $\alpha_1=\beta_1=R_1=A(\xi)=B(\xi)=C(\xi)=0$.

In the case when either $\Delta_2(K)=0$ or $\Delta_2(v)=0$, resonance occurs between the cylinders because of the wave transmission through the porous barrier.

2.2.2. *Case of infinite depth with single permeable cylinder*

When there is no inner wall, the general boundary-value problem reduces to the solution of the function $\phi(r,y)$ satisfying the relations (2.18), (2.2), (2.3), (2.7), (2.20) and (2.21). In this case, its general representation is given by

$$\phi(r,y)=\begin{cases}\alpha_1J_0(Kr)\mathrm{e}^{-Ky}+\alpha_2J_0(vr)l(y)+\displaystyle\int_0^\infty A(\xi)I_0(\xi r)L(\xi,y)\mathrm{d}\xi, & 0<r<b,\\ A_1H_0^{(2)}(Kr)\mathrm{e}^{-Ky}+R_1H_0^{(1)}(Kr)\mathrm{e}^{-Ky}+A_2H_0^{(2)}(vr)l(y)+R_2H_0^{(1)}(vr)l(y) & \\ \qquad+\displaystyle\int_0^\infty C(\xi)K_0(\xi r)L(\xi,y)\mathrm{d}\xi, & r>b,\end{cases}\tag{2.26}$$

where $\alpha_1, \alpha_2, R_1, R_2, A(\xi)$ and $C(\xi)$ are unknowns to be determined and $l(y), L(\xi, y)$ have their usual meaning as in Subsection 2.1.

The unknowns can be determined by using the continuity of the velocity across the porous wall and the condition (2.20) on the porous wall and the particular cases of physical interest are as follow.

Wavemaker problem. This is the case of wave trapping and we must have $u_2(y) \neq 0, A_1 = 0$ and $A_2 = 0$. The unknown constants, under the constraints $J_1(Kb) \neq 0$ and $J_1(vb) \neq 0$, in the solution (2.26) are given by

$$\alpha_1 = R_1 \frac{H_1^{(1)}(Kb)}{J_1(Kb)}, \quad \alpha_2 = R_2 \frac{H_1^{(1)}(vb)}{J_1(vb)}$$

with

$$R_1 = -\frac{2K J_1(Kb) e^{2Kh} a_2(K)}{[\rho_2 + \rho_1(e^{2Kh} - 1)]}$$

and

$$R_2 = -\frac{2v J_1(vb) b_2(v)}{[\rho_2 + 2v\rho_1 \int_0^h g^2(y) dy]}.$$

Waves generated with frequencies K or v satisfying either $J_1(Kb) = 0$ or $J_1(vb) = 0$, get trapped between the cylinders and causing a tranquil zone outside the cylinder.

Wave scattering. This case represents the resonance of incident waves. For free-surface wave incidence, we must have $u_2(y) = 0, A_1 = 1$ and $A_2 = 0$. The unknown coefficients, under the constraint that $J_1(Kb) \neq 0$, in the solution (2.26) reduce to

$$\alpha_1 = R_1 \frac{H_1^{(1)}(Kb)}{J_1(Kb)} + \frac{H_1^{(2)}(Kb)}{J_1(Kb)} \quad \text{with} \quad R_1 = \frac{2G - \pi K^2 b H_1^{(2)}(Kb) J_1(Kb)}{2G + \pi K^2 b H_1^{(1)}(Kb) J_1(Kb)}.$$

For the interfacial wave incidence, $u_2(y) = 0, A_1 = 0$ and $A_2 = 1$ must be taken and the unknowns, under the constraint that $J_1(vb) \neq 0$, become

$$\alpha_2 = R_2 \frac{H_1^{(1)}(vb)}{J_1(vb)} + \frac{H_1^{(2)}(vb)}{J_1(vb)} \quad \text{with} \quad R_2 = \frac{2G - \pi v^2 b H_1^{(2)}(vb) J_1(vb)}{2G + \pi v^2 b H_1^{(1)}(vb) J_1(vb)}.$$

Note that R_1 and R_2 are derived with G replaced by $-G$ above because of the flow considerations and the above linear analysis predicts that resonance occurs inside the porous cylinder by the transmitted surface or interfacial wave with a frequency satisfying either $J_1(Kb) = 0$ or $J_1(vb) = 0$.

2.2.3. *Case of finite depth with two co-axial cylinders*

The general solution ϕ satisfying the relations (2.2), (2.3), (2.8), (2.18–2.20) and (2.22) can be represented as

$$\phi(r, y) = \begin{cases} \sum_{j=1}^{2} \left[\alpha_j J_0(m_j r) + \beta_j H_0^{(1)}(m_j r) \right] f_0(m_j, y) + \sum_{n=1}^{\infty} \left[B_n I_0(p_n r) + C_n K_0(p_n r) \right] f_n(p_n, y), \\ \qquad 0 < a < r < b, \quad 0 < y < \infty, \\ \sum_{j=1}^{2} \left[A_j H_0^{(2)}(m_j r) + R_j H_0^{(1)}(m_j r) \right] f_0(m_j, y) + \sum_{n=1}^{\infty} D_n K_0(p_n r) f_n(p_n, y), \\ \qquad b < r < \infty, \quad 0 < y < \infty, \end{cases} \tag{2.27}$$

where $\alpha_j, \beta_j, R_j, j=1,2$ and $B_n, C_n, D_n, n=1,2,3\ldots$ are the unknowns to be determined and the functions $f_0(m_j, y), j=1,2$ and $f_n(p_n, y), n=1,2,3\ldots$ are defined in Subsection 2.1.

Like in the infinite-depth case, we briefly mention how the unknowns in the relation (2.27) are determined. Using the continuity of the velocity across $r=b$, the condition (2.20) on the porous barrier at $r=b$ and the condition (2.19) on the inner wall $r=a$, we can derive three equations. Applying the orthogonality of the functions involved and after few calculations, all the unknown can be derived in terms of known functions.

Now, we explain the particular problems of physical interest one by one via the unknowns associated with the wave parts in the relation (2.27).

Waves generated by the inner cylinder while the outer cylinder is kept fixed. In this case $u_1(y)\neq 0, u_2(y)=0, A_1=0$ and $A_2=0$ and the unknowns, under the restriction that $\Delta_2(m_j)\neq 0, j=1,2$, are determined as

$$\alpha_i = R_i\frac{H_1^1(m_i a)H_1^1(m_i b)}{\Delta_2(m_i)} - \frac{\left\langle u_1(y), f_0(m_i, y)\right\rangle_2 H_1^1(m_i b)}{m_i M_0(m_i)\Delta_2(m_i)}, \quad i=1,2,$$

$$\beta_i = -R_i\frac{J_1(m_i a)H_1^1(m_i b)}{\Delta_2(m_i)} - \frac{\left\langle u_1(y), f_0(m_i, y)\right\rangle_2 J_1(m_i b)}{m_i M_0(m_i)\Delta_2(m_i)}, \quad i=1,2,$$

and

$$R_i = \frac{2G\left\langle u_1(y), f_0(m_i, y)\right\rangle_2}{m_i M_0(m_i)\Delta_3(m_i)}, \quad i=1,2.$$

Clearly, the above analysis predicts that wave resonance occurs between the cylinders for the frequencies with $\Delta_2(m_j)=0, j=1,2$.

Waves generated by the outer cylinder while the inner cylinder is kept fixed. This is the case where the generated wave get trapped inside the cylinder and here $u_1(y)=0, u_2(y)\neq 0, A_1=0$ and $A_2=0$. The unknowns, under the restriction that $\Delta_2(m_j)\neq 0, j=1,2$, are determined as

$$\alpha_i = R_i\frac{H_1^1(m_i a)H_1^1(m_i b)}{\Delta_2(m_i)}, \quad i=1,2, \quad \beta_i = -R_i\frac{J_1(m_i a)H_1^1(m_i b)}{\Delta_2(m_i)}, \quad i=1,2,$$

and

$$R_i = -\frac{\pi b m_i\Delta_2(m_i)\left\langle u_2(y), f_0(m_i, y)\right\rangle_2}{M_0(m_i)\Delta_3(m_i)}, i=1,2, \quad B_n = D_n\frac{K_1(p_n a)K_1(p_n b)}{\Delta_1(p_n)}, \quad n=1,2,3\ldots$$

It may be remarked that the generated wave with the frequencies satisfying $\Delta_2(m_j)=0, j=1,2$ get trapped between the cylinders, while it maintains a calmer region outside the cylinder.

Scattering of water waves when the two co-axial cylinders are kept fixed. In this case $u_1(y)=0, u_2(y)=0$ and, if the incident wave is the free-surface mode, we must have $A_1=1$ and $A_2=0$ and the unknowns, under the restriction that $\Delta_2(m_1)\neq 0$, are determined as

$$\alpha_1 = R_1\frac{H_1^1(m_1 a)H_1^1(m_1 b)}{\Delta_2(m_1)} + \frac{H_1^1(m_1 a)H_1^2(m_1 b)}{\Delta_2(m_1)},$$
$$\beta_1 = -R_1\frac{J_1(m_1 a)H_1^1(m_1 b)}{\Delta_2(m_1)} - \frac{J_1(m_1 a)H_1^2(m_1 b)}{\Delta_2(m_1)},$$

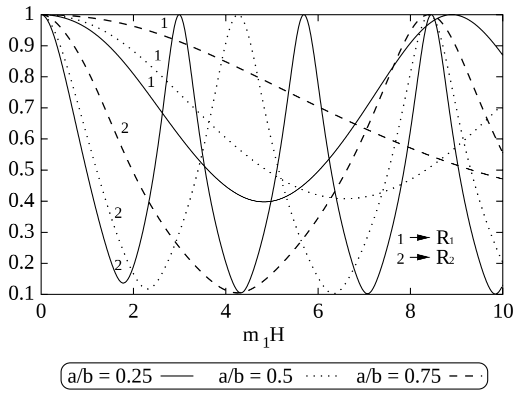

Figure 8. Reflection coefficients vs. $m_1 H$ in case of two co-axial cylinders for different a/b values with $h/H = 0.5$, $s = 0.5$, $G = 1 + 2.0$ and $b/H = 0.75$.

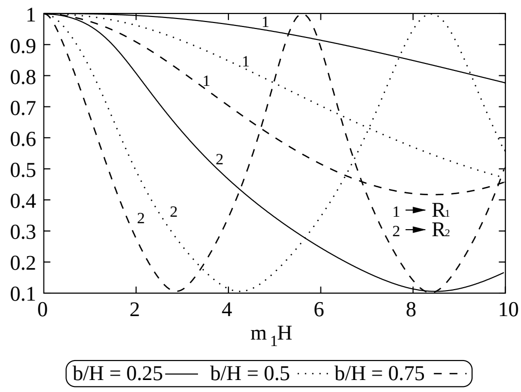

Figure 9. Reflection coefficients vs. $m_1 H$ in case of two co-axial cylinders for different b/H values with $h/H = 0.5$, $s = 0.5$, $G = 1 + 2.0$ and $a/b = 0.75$.

$$R_1 = \frac{2GH_1^{(2)}(m_1 a) + \pi b m_1^2 \Delta_2(m_1) H_1^{(2)}(m_1 b)}{2GH_1^{(1)}(m_1 a) + \pi b m_1^2 \Delta_2(m_1) H_1^{(1)}(m_1 b)}$$

and $\alpha_2 = \beta_2 = R_2 = 0$.

For the interfacial incident wave, $A_1 = 0$ and $A_2 = 1$ must be taken and in this case the unknowns, under the restriction that $\Delta_2(m_2) \neq 0$, are given by

$$\alpha_2 = R_2 \frac{H_1^1(m_2 a) H_1^1(m_2 b)}{\Delta_2(m_2)} + \frac{H_1^1(m_2 a) H_1^2(m_2 b)}{\Delta_2(m_2)},$$

$$\beta_2 = -R_2 \frac{J_1(m_2 a) H_1^1(m_2 b)}{\Delta_2(m_2)} - \frac{J_1(m_2 a) H_1^2(m_2 b)}{\Delta_2(m_2)},$$

$$R_2 = -\frac{2GH_1^{(2)}(m_2 a) + \pi b m_2^2 \Delta_2(m_2) H_1^{(2)}(m_2 b)}{2GH_1^{(1)}(m_2 a) + \pi b m_2^2 \Delta_2(m_2) H_1^{(1)}(m_2 b)}$$

and $\alpha_1 = \beta_1 = R_1 = 0$.

Note that, when $G = 0$, and for large values of G, the reflection coefficients in SM and IM become 1, which corresponds to the case of full reflection, as expected. And when $\Delta_2(m_j) = 0$, $j = 1, 2$, m_j, $j = 1, 2$ represent the resonance frequencies.

In order to understand the general behavior of the wave reflection by the porous cylinder, reflection coefficients are plotted against $m_1 H$ for various values of a/b in Figure 8 for both the cases of SM and IM. The pattern of the reflection coefficient is observed to be similar to the one in [21]. The occurrence of a minimum and a maximum in the reflection coefficient is more pronounced for wave motion in IM as compared to that in SM, showing the significance of the internal waves. In addition, with an increase in a/b, the occurrence of maxima and minima decreases in both the cases of SM and IM. A similar phenomenon in the reflection coefficient is observed in both the cases of SM and IM for different values of b/H, as is seen in Figure 9.

2.2.4. *Case of finite depth with single permeable cylinder*
When there is no inner wall, the general representation for the potential ϕ satisfying the relations (2.18), (2.2), (2.3), (2.8), (2.20), and (2.22) is given by

$$\phi(r,y)=\begin{cases}\sum_{j=1}^{2}\alpha_j J_0(m_j r) f_0(m_j,y)+\sum_{n=1}^{\infty} B_n I_0(p_n r) f_n(p_n,y), \quad 0<r<b, \quad 0<y<\infty,\\ \sum_{j=1}^{2}\Big[A_j H_0^{(2)}(m_j r)+R_j H_0^{(1)}(m_j r)\Big] f_0(m_j,y)+\sum_{n=1}^{\infty} C_n K_0(p_n r) f_n(p_n,y),\\ \qquad\qquad b<r<\infty, \quad 0<y<\infty,\end{cases} \tag{2.28}$$

where $\alpha_1,\alpha_2,R_1,R_2,B_n,n=1,2,3\ldots$ and $C_n,n=1,2,3\ldots$ are the unknowns to be determined.

Again, here too, using the continuity of the velocity across $r=b$, the condition (2.20) on the porous barrier at $r=b$ and the orthogonal property of the functions, we can easily determine the unknown constants in the relation (2.28).
Wavemaker problem. In this case, we must have $u_2(y)\neq 0$, $A_1=0$ and $A_2=0$ and under the restriction that $J_1(m_j)\neq 0$, $j=1,2$ the unknown coefficients in the solution (2.28) reduces to

$$\alpha_i=R_i\frac{H_1^{(1)}(m_i b)}{J_1(m_i b)}, \quad i=1,2$$

with

$$R_i=-\frac{\pi b m_1\Big\langle u_2(y), f_0(m_i,y)\Big\rangle_2 J_1(m_i b)}{M_0(m_i)\Big[2G-\pi b m_i^2 H_1^{(1)}(m_i b) J_1(m_i b)\Big]}, \quad i=1,2.$$

For a porous-wavemaker, the generated wave gets trapped inside the cylinder while it maintains a tranquil zone outside the cylinder for waves with frequencies satisfying $J_1(m_j)=0$, $j=1,2$.
Wave scattering. In this case, we must have $u_2(y)=0$, $A_1=1$ and $A_2=0$ for the free-surface incident wave. The unknown coefficients in the solution (2.28), under the restriction that $J_1(m_1)\neq 0$, are given by

$$\alpha_1=R_1\frac{H_1^{(1)}(m_1 b)}{J_1(m_1 b)}+\frac{H_1^{(2)}(m_1 b)}{J_1(m_1 b)} \quad \text{with} \quad R_1=\frac{2G-\pi b m_1^2 H_1^{(2)}(m_1 b) J_1(m_1 b)}{2G+\pi b m_1^2 H_1^{(1)}(m_1 b) J_1(m_1 b)}$$

and $\alpha_2=R_2=0$.

For the interfacial wave incidence, $u_2(y)=0$, $A_1=1$ and $A_2=0$ should be taken and in this case the unknowns, under the restriction that $J_1(m_2)\neq 0$, are obtained as

$$\alpha_2=R_2\frac{H_1^{(1)}(m_2 b)}{J_1(m_2 b)}+\frac{H_1^{(2)}(m_2 b)}{J_1(m_2 b)} \quad \text{with} \quad R_2=\frac{2G-\pi b m_2^2 H_1^{(2)}(m_2 b) J_1(m_2 b)}{2G+\pi b m_2^2 H_1^{(1)}(m_2 b) J_1(m_2 b)}$$

and $\alpha_1=R_1=0$.

Note that R_1,R_2 are derived with G replaced by $-G$ above because of the flow considerations and note that there is a loss of energy when the porous parameter $G\neq 0$. In the case of $J_1(m_j b)=0$, $j=1,2$, resonance will occur inside the cylinder for all values of the porosity. The resonance condition within the porous cylinder is similar to that of wave resonance within a circular tank having rigid boundary. When $G=0$, the wave gets fully reflected by the rigid wall of the cylinder in both SM and IM. The reflection coefficient pattern in Figures 10 is observed to be similar to that of the two co-axial cylinders.

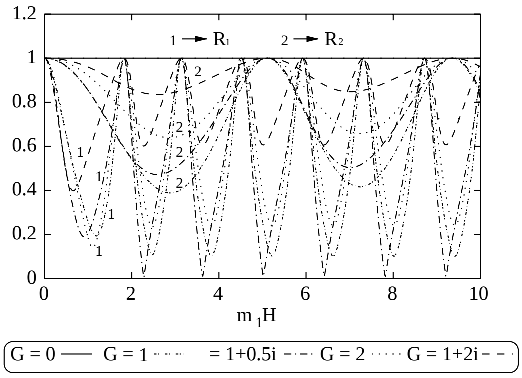

Figure 10. Reflection coefficients vs. m_1H in case of single cylinder for different G values with $h/H=0.5$, $s=0.5$ and $b/H=0.75$.

3. Derivation of line source potentials

The generation of surface and interfacial oblique water waves involves the consideration of different types of singularities in the fluid under consideration. When these waves are generated by a body present in the fluid, the resulting motion can be described by a series of line singularities placed within the body. Under the action of gravity, the time-harmonic irrotational motion of a two-layer incompressible and inviscid fluid having different but constant densities is considered.

The symmetric oblique-wave source potential is essentially the function $\phi(x, y|x_0, y_0)$ satisfying the relation (2.1) in the fluid region except at the source point (x_0, y_0), where $0<y_0<\infty$ for infinite depth and $0<y_0<H$ for finite depth and the relations (2.2) and (2.3).

The function $\phi(x, y|x_0, y_0)$ satisfies the relation (2.7) for infinite depth, the relation (2.8) for finite depth.

At the source point, *i.e.*, as $(x, y)\to(x_0, y_0)$,

$$\phi\sim-\frac{1}{2\pi}K_0(l_0r)\sim\frac{1}{2\pi}\log(l_0r), \tag{3.1}$$

where K_0 is a modified Bessel function and $r=\sqrt{(x-x_0)^2+(y-y_0)^2}$.

Also,

$$\phi(x, y)\sim\begin{cases} A_1\mathrm{e}^{\mathrm{i}\mu|x-x_0|-Ky}+A_2\mathrm{e}^{\mathrm{i}\beta|x-x_0|}l(y), & \text{for infinite depth} \\ \displaystyle\sum_{j=1}^{2}A_j\mathrm{e}^{\mathrm{i}\mu_j|x-x_0|}f_0(m_j, y), & \text{for finite depth} \end{cases} \quad \text{as } |x-x_0|\to\infty, \tag{3.2}$$

where A_j, $j=1,2$ are unknown constants to be determined.

The above problem of finding the symmetric wave-source potential can be recast into the boundary-value problem of finding the potential function $\phi(x, y)$ in the quarter plane $x>x_0, y>0$ satisfying the relation (2.1) and the relations (2.2), (2.3), (2.7) or (2.8) and (3.2) depending on the depth of the water and

$$\frac{\partial\phi}{\partial x}=\frac{1}{2}\delta(y-y_0) \text{ on } x=x_0, \tag{3.3}$$

where $0\le y_0<\infty$ for infinite depth and $0\le y_0\le H$ for finite depth.

The above condition (3.3) follows upon differentiating the condition $\phi \sim \frac{1}{2\pi}\log(l_0 r)$ as $r \to 0$ and making use of the generalized identity

$$\lim_{x\to x_0} \frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} = \pi\delta(y-y_0).$$

The solutions for the boundary-value problem in infinite and finite depths are derived as follows:

3.1. Case of infinite depth

The general form of the potential function $\phi(x, y)$ is represented by

$$\phi(x,y) = A_1 e^{i\mu(x-x_0)-Ky} + A_2 e^{i\beta(x-x_0)} l(y) + \int_0^\infty A(\xi) L(\xi, y) e^{-(x-x_0)\sqrt{\xi^2+l_0^2}} d\xi, \tag{3.4}$$

with $A(\xi)$ being an unknown function yet to be determined.

Using the condition (3.3) and the property of the delta function as given by

$$\int_0^\infty \delta(y-y_0) f(y) dy = f(y_0) S(y_0) \quad \text{where} \quad S(y_0) = \begin{cases} 1, & \text{if } y_0 > 0 \\ \frac{1}{2}, & \text{if } y_0 = 0, \end{cases}$$

we derive the unknowns in the expansion (3.4) which are given by

$$A_1 = \begin{cases} -\dfrac{i\rho_1 K e^{2Kh} e^{-Ky_0} S(y_0)}{\mu\left[\rho_2 + \rho_1(e^{2Kh}-1)\right]}, & \text{if } 0 \le y_0 < h \\ -\dfrac{i\rho_2 K e^{2Kh} e^{-Ky_0}}{\mu\left[\rho_2 + \rho_1(e^{2Kh}-1)\right]}, & \text{if } h < y_0 < \infty \\ -\dfrac{i(\rho_1+\rho_2) K e^{Kh}}{2\mu\left[\rho_2 + \rho_1(e^{2Kh}-1)\right]}, & \text{if } y_0 = h, \end{cases}$$

$$A_2 = \begin{cases} -\dfrac{i\rho_1 v g(y_0) S(y_0)}{\beta\left[\rho_2 + 2\rho_1 v \int_0^h g^2(y) dy\right]}, & \text{if } 0 \le y_0 < h \\ -\dfrac{i\rho_2 v e^{v(h-y_0)}}{\beta\left[\rho_2 + 2\rho_1 v \int_0^h g^2(y) dy\right]}, & \text{if } h < y_0 < \infty \\ -\dfrac{iv[\rho_1 g(h) + \rho_2]}{2\beta\left[\rho_2 + 2\rho_1 v \int_0^h g^2(y) dy\right]}, & \text{if } y_0 = h, \end{cases}$$

and

$$A(\xi) = \begin{cases} -\dfrac{\rho_1\rho_2 K[\xi\cos\xi y_0 - K\sin\xi y_0] S(y_0)}{\pi\sqrt{\xi^2+l_0^2}\,\mathcal{D}_0(\xi)(\xi^2+K^2)}, & \text{if } 0 \le y_0 < h \\ -\dfrac{\rho_2^2 K(\xi\cos\xi y_0 - K\sin\xi y_0) + \rho_2(\rho_2-\rho_1)(\xi^2+K^2)\sin\xi h\cos\xi(y_0-h)}{\pi\sqrt{\xi^2+l_0^2}\,\mathcal{D}_0(\xi)(\xi^2+K^2)}, & \text{if } h < y_0 < \infty \\ -\dfrac{K\rho_2(\rho_1+\rho_2)(\xi\cos\xi h - K\sin\xi h) + \rho_2(\rho_2-\rho_1)(\xi^2+K^2)\sin\xi h}{2\pi\sqrt{\xi^2+l_0^2}\,\mathcal{D}_0(\xi)(\xi^2+K^2)}, & \text{if } y_0 = h. \end{cases}$$

3.2. Case of finite depth

The general form of the potential function, in this case, is given by

$$\phi(x, y) = \sum_{j=1}^{2} A_j e^{i\mu_j (x-x_0)} f_0(m_j, y) + \sum_{n=1}^{\infty} B_n f_n(p_n, y) e^{-(x-x_0)\sqrt{p_n^2+l_0^2}}, \tag{3.5}$$

where $B_n, n=1,2,3\ldots$ are unknown constants to be determined.

Utilizing the condition (3.3), the unknowns in the relation (3.5) are obtained as

$$A_j = \begin{cases} -\dfrac{i\rho_1 \sinh m_j(h-H)}{2\mu_j M_0(m_j)}[m_j \cosh m_j y_0 - K \sinh m_j y_0] S(y_0), & \text{if } 0 \le y_0 < h \\ -\dfrac{i\rho_2 [m_j \sinh m_j h - K \cosh m_j h]}{2\mu_j M_0(m_j)} \cosh m_j (y_0 - H) S(H - y_0), & \text{if } h < y_0 \le H \\ & j=1,2 \\ -\dfrac{i\rho_1 [m_j \cosh m_j h - K \sinh m_j h] \sinh m_j (h-H)}{4\mu_j M_0(m_j)} & \\ -\dfrac{i\rho_2 [m_j \sinh m_j h - K \cosh m_j h] \cosh m_j (h-H)}{4\mu_j M_0(m_j)}, & \text{if } y_0 = h, \end{cases}$$

and

$$B_n = \begin{cases} -\dfrac{\rho_1 \sin p_n(h-H)}{2M_n(p_n)\sqrt{p_n^2+l_0^2}}[p_n \cos p_n y_0 - K \sin p_n y_0] S(y_0), & \text{if } 0 \le y_0 < h \\ -\dfrac{\rho_2 [p_n \sin p_n h + K \cos p_n h]}{2M_n(p_n)\sqrt{p_n^2+l_0^2}} \cos p_n (y_0 - H) S(H - y_0), & \text{if } h < y_0 \le H \\ & n=1,2,3\ldots \\ -\dfrac{\rho_1 [p_n \cos p_n h - K \sin p_n h] \sin p_n (h-H)}{4M_n(p_n)\sqrt{p_n^2+l_0^2})} & \\ -\dfrac{\rho_2 [p_n \sin p_n h + K \cos p_n h] \cos p_n (h-H)}{4M_n(p_n)\sqrt{p_n^2+l_0^2}}, & \text{if } y_0 = h, \end{cases}$$

We remark here that, by allowing $h \to 0$ and $\rho_1 \to \rho_2$ simultaneously, the source potentials in either of the depths are reduced to the known potentials in a single layer of fluid. Also, the strength of the source is doubled if a source point appears either at the free surface or at the bottom boundary in the case of finite depth.

3.3. Multipole linesource wave potentials

The boundary-value problem for the symmetric and antisymmetric multipole wave potentials $\phi_s(x, y|x_0, y_0), \phi_a(x, y|x_0, y_0)$ is the same as the one for the source potentials described above, except that the condition (3.1) is replaced by $\phi_s \sim \dfrac{1}{2\pi}\dfrac{\cos n\theta}{r^n}$, $\phi_a \sim \dfrac{1}{2\pi}\dfrac{\sin n\theta}{r^n}$, respectively, as $r \to 0$ for $n = 1,2,3\ldots$, where $x - x_0 = r\cos\theta$, $y - y_0 = r\sin\theta$, with $0 < \theta < \pi$.

Using the representations

$$\log\ r = \int_0^\infty \frac{1}{\xi}(e^{-\xi} - e^{-\xi|y-y_0|}) \cos \xi(x - x_0) d\xi, \quad y, y_0 > 0,$$

$$\frac{\cos n\theta}{r^n} = \frac{[\text{sgn}(y-y_0)]^n}{(n-1)!}\int_0^\infty \xi^{n-1} e^{-\xi|y-y_0|} \cos \xi(x-x_0) d\xi, \quad y, y_0 > 0, \ n = 1,2,3\ldots,$$

and

$$\frac{\sin n\theta}{r^n}=\frac{[\mathrm{sgn}(y-y_0)]^{n+1}}{(n-1)!}\int_0^\infty \xi^{n-1}\mathrm{e}^{-\xi|y-y_0|}\cos\xi(x-x_0)\mathrm{d}\xi,\quad y,y_0>0,\ \ n=1,2,3\ldots,$$

where sgn is the sign function, we observe that $\frac{\cos n\theta}{r^n}=-\frac{1}{(n-1)!}\frac{\partial^n\log r}{\partial y_0^n}$, and $\frac{\sin n\theta}{r^n}=-\frac{1}{(n-1)!}\frac{\partial^n\log r}{\partial x_0\partial y_0^{n-1}}$, $n=1,2,3\ldots$, Therefore, the expansions, while existing, for the symmetric and antisymmetric multipole wave potentials $\phi_s(x,y|x_0,y_0),\phi_a(x,y|x_0,y_0)$ can be represented as

$$\phi_s(x,y|x_0,y_0)=-\frac{1}{(n-1)!}\frac{\partial^n\phi}{\partial y_0^n},\quad \phi_a(x,y|x_0,y_0)=-\frac{1}{(n-1)!}\frac{\partial^n\phi}{\partial x_0\partial y_0^{n-1}}\ \text{ for } n=1,2,3\ldots,$$

where ϕ is the wave-source potential given by the relations (3.4), (3.5) for infinite and finite depths, respectively.

4. Conclusions

The expansion formulae for the velocity potentials in a two-layer fluid domain, having a free surface and an interface, have been derived to analyse wave motion past a porous structure, in both the cases of water of finite and infinite depths under the assumptions of the linearised theory of water waves. In order to determine the unknown constants in the expansion formulae for wave motion in a two-layer fluid, a more general type of orthogonal relations has been utilized which are a generalization of the one used for wave problems in a single fluid domain of homogeneous density. As an application of the expansion formulae, oblique water wave radiation and scattering by thin porous structures has been analyzed for both the cases of finite and infinite water depths. Due to the presence of a free surface and interface, there are two wave modes of propagation, which are referred to as waves in surface mode and interface mode. Also, the reflection and transmission coefficients of wave scattering by a porous barrier and the far-field wave amplitude of the porous wavemaker have been derived in explicit form for both surface and interface modes in a two-layer fluid. Numerical results for the reflection and transmission coefficients and the surface elevations at the surface and the interface, along with the amplitude of the force acting on the structure have been plotted and analyzed for specific cases in order to understand the role of waves propagating both at the free surface and the interface. The axisymmetric wave motion due to co-axial permeable or/and impermeable cylinders has been analyzed and various analytical results have been presented for the generation and scattering of the two wave modes present in a two-layer fluid. The condition of resonance within(between) the cylinder(s) was explained and the phenomenon of wave trapping between the two co-axial cylinders analyzed. It is observed that the interfacial waves resist the porous structure significantly. The present study will be useful in future design of coastal and offshore structures in a stratified ocean which can be modelled as a two-layer fluid having a free surface. Finally, the wave source and multipole source potentials for oblique surface water waves have been derived, which are useful particularly in semi-analytical studies of wave-related problems involving submerged cylinders. By the present expansion formulae of the velocity potentials and the associated orthogonal relations, a large class of problems in a two-layer fluid having a free surface can be simplified to a large extent. In addition, the analysis of the present study can be extended to investigate a large class of problems in the area of fluid-structure interactions arising its broad areas of mathematical physics and engineering.

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